Warwick Mathematics Exchange

MA131/MA137

Analysis I & II

2022, June 8th

Desync, aka The Big Ree

Contents

Introduction

Analysis is the study of limits and related concepts - notably, sequences, series, differentiation and integration. Due to the proofy nature of this module, I will not be including a condensed list of calculations as was included for MA106 and MA133. I have tried to include worked examples where possible, but, to be honest, most of this module is just memorising proofs, and not many examples are particularly helpful.

This document is intended to broadly cover all the topics within the (Mathematical) Analysis II modules. A section recapping Analysis I is included, but only results will be stated, with any proofs and much intuition omitted. All such results will be written in a highly condensed and symbolic form.

Disclaimer: This document was made by a first year student who stopped going to lectures for the second half of this module after week 2. I make *absolutely no guarantee* that this document is complete nor without error. In particular, any content covered exclusively in lectures (if any) will not be recorded here. Additionally, I do not take Mathematical Analysis, so the accuracy of the MA137 sections may be impacted. This document was written at the end of the 2022 academic year, so any changes in the course since then may not be accurately reflected.

Notes on formatting

New terminology will be introduced in *italics* when used for the first time. Named theorems will also be introduced in *italics*. Important points will be **bold**. Common mistakes will be underlined. The latter two classifications are under my interpretation. YMMV.

Content exclusive to MA131 will be outlined in the margins like this.

The table of contents above, and any inline references are all hyperlinked for your convenience.

History

First Edition: 2022-06-04*
Current Edition: 2022-06-10

Authors

This document was written by R.J. Kit L., a maths student. I am not otherwise affiliated with the university, and cannot help you with related matters.

Please send me a PM on Discord @Desync#6290, a message in the WMX server, or an email to Warwick.Mathematics.Exchange@gmail.com for any corrections. (If this document somehow manages to persist for more than a few years, these contact details might be out of date, depending on the maintainers. Please check the most recently updated version you can find.)

If you found this guide helpful and want to support me, you can buy me a coffee!

(Direct link for if hyperlinks are not supported on your device/reader: ko-fi.com/desync.)

*Storing dates in big-endian format is clearly the superior option, as sorting dates lexicographically will also sort dates chronologically, which is a property that little and middle-endian date formats do not share. See ISO-8601 for more details. This footnote was made by the computer science gang.

1 Analysis I

1.1 Inequalities

$x < y \Leftrightarrow x + a < y + a$.

$a > 0, x < y \Leftrightarrow ax < ay$.

$a < 0, x < y \Leftrightarrow ax > ay$.

$x < y, y < z \Rightarrow x < z$.

Power Rule: If x,$y \in \mathbb{R}^+$ then for each $n \in \mathbb{N}$, $x < y$ if and only if $x^n < y^n$.

Idempotency of Modulus Function: $||x|| = |x|$

$|xy| = |x||y|$

Interval Property: If $x \in \mathbb{R}$ and $r \in \mathbb{R}^+$, then $|x| < r$ if and only if $-r < x < r$. *Corollary*: If x,$a \in \mathbb{R}$ and $r \in \mathbb{R}^+$, then $|x - a| < r$ if and only if $a - r < x < a + r$.

Triangle Inequality: For all x,$y \in \mathbb{R}$, $|x + y| \leq |x| + |y|$.

1.2 Sequences

Definition 1.1. A *sequence* is a function $a : \mathbb{N} \to \mathbb{R}$, usually denoted as $a(n) = a_n$.

We will use the convention that a sequence starts at $n = 1$. When we talk about a sequence as a whole, we wrap the function in brackets as follows: (a_n).

Definition 1.2. A sequence, (a_n), is:

- *strictly increasing* if $a_{n+1} > a_n \,\forall n$;
- *increasing* if $a_{n+1} \geq a_n \,\forall n$;
- *strictly decreasing* if $a_{n+1} < a_n \,\forall n$;
- *decreasing* if $a_{n+1} \leq a_n \,\forall n$;
- *monotonic* if a_n is increasing or decreasing or both (constant);

Definition 1.3. A sequence, (a_n) is:

- *bounded above* if $\exists M \in \mathbb{R}$ such that $a_n \leq M \,\forall\, n$. M is then an *upper bound* of the sequence;
- *bounded below* if $\exists m \in \mathbb{R}$ such that $a_n \geq m \,\forall\, n$. m is then a *lower bound* of the sequence;
- *bounded* if it is bounded above **and** below.

Upper and lower bounds are not unique. For example, if M is an upper bound of (a_n), then $M + 1$ is clearly also an upper bound.

Every increasing sequence is bounded below by its first term.

Every decreasing sequence is bounded above by its first term.

Definition 1.4. A sequence (a_n) *tends to infinity* if for every $C > 0$, there exists $N \geq 1$ such that $a_n > C$ for all $n > N$.

Definition 1.5. A sequence (a_n) *tends to minus infinity* if for every $C > 0$, there exists $N \geq 1$ such that $a_n < -C$ for all $n > N$.

Let (a_n) and (b_n) be sequences such that $b_n \geq a_n \,\forall\, n$, and suppose that $a_n \to \infty$. Then, $b_n \to \infty$.

Suppose (a_n) and (b_n) tend to infinity, then,

- $a_n + b_n \to \infty$
- $a_n b_n \to \infty$
- $ca_n \to \infty$ if $c > 0$
- $ca_n b_n \to -\infty$ if $c < 0$

Definition 1.6. A sequence, (a_n) *converges to* or *tends to* a if $\forall \epsilon > 0 \, \exists N \in \mathbb{N}$ such that $|a_n - a| < \epsilon \, \forall n > N$.

If such an a exists, the sequence is *convergent*, and a is the *limit* of the sequence. Otherwise, the sequence is *divergent*. Note that sequences diverge to (minus) infinity.

Uniqueness of Limits: A sequence can converge to at most one limit.

Every convergent sequence is bounded.

Definition 1.7. A sequence (a_n) is a *null sequence* if $(a_n) \to 0$.

Let a,$b \in \mathbb{R}$, and $(a_n) \to a$ and $(b_n) \to b$ be convergent sequences. Then,

- $ca_n + db_n \to ca + db$.
- $a_n b_n \to ab$.
- $\frac{a_n}{b_n} \to \frac{a}{b}$, provided $b \neq 0$.

These properties are known as *algebra of limits*.

Absolute Value Rule: $(a_n) \to 0$ if and only if $(|a_n|) \to 0$.

Sandwich Theorem for Sequences: Suppose $(a_n) \to l$ and $(b_n) \to l$. If $a_n \leq c_n \leq b_n$, then, $(c_n) \to l$.

Definition 1.8. A sequence (a_n) satisfies a certain property *eventually* if there exists an integer $n \geq 0$ such that the sequence (a_{N+n}) satisfies that property.

If a sequence is eventually bounded, it is bounded.

Shift Rule: Let $N \geq 0$ be an integer, and let (a_n) be a sequence. Then, $(a_n) \to a$ if and only if $(a_{N+n}) \to a$.

Sandwich Theorem with Shift Rule: Suppose $(a_n) \to l$ and $(b_n) \to l$. If eventually $a_n \leq c_n \leq b_n$ then $(c_n) \to l$.

Inequality Rule: Suppose $(a_n) \to a$ and $(b_n) \to b$. If eventually $a_n \leq b_n$ then $a \leq b$.

Closed Interval Rule: Suppose $(a_n) \to a$. If eventually $A \leq a_n \leq B$, then $A \leq a \leq B$.

If the inequalities above are strict, the lemma does not hold true; limits can escape open intervals as far as the endpoints. For example, $0 < \frac{n}{n+1} < 1 \, \forall n$, but $\frac{n}{n+1} \to 1$.

Definition 1.9. A *subsequence* of (a_n) is a sequence of the form (a_{n_i}), where (n_i) is a strictly increasing subsequence of natural numbers.

If (a_n) is convergent if and only if every subsequence is convergent.

If (a_n) diverges to $\pm\infty$ then every subsequence diverges to $\pm\infty$.

If (a_n) is bounded above (below), then every subsequence is bounded above (below).

If (n_i) is a strictly increasing sequence of natural numbers, then for all $i \geq 1$, $n_i \geq i$.

Monotonic Subsequence Theorem: Every sequence contains a monotonic subsequence.

Definition 1.10. a_f is a *floor term* of (a_n) if $a_n \geq a_f$ for all $n \geq f$.

Each floor term is eventually a lower bound.

Ceiling terms are defined similarly.

1.3 Roots & Powers

Bernoulli's Inequality: If $x > -1$ and $n \in \mathbb{N}$, then $(1+x)^n \geq 1 + nx$.

Binomial Theorem: For all $x,y \in \mathbb{R}$, $(x+y)^n = \sum_{k=0}^{n} \binom{n}{k} x^k y^{n-k}$, where $\binom{n}{k} = \frac{n!}{k!(n-k)!}$.

If $x > 0$, then $(x^{\frac{1}{n}} \to 1)$.

$(n^{\frac{1}{n}}) \to 1$.

Ratio Lemma Let (a_n) be a sequence such that $a_n > 0$ for all n. Suppose $0 < l < 1$ and $\frac{a_{n+1}}{a_n} \leq l$ eventually. Then $(a_n) \to 0$. *Corollary* Let (a_n) be a sequence such that $a_n > 0$ for all n. If $\left(\frac{a_{n+1}}{a_n}\right) \to a$ and $0 \leq a < 1$ then $(a_n) \to 0$.

1.4 Completeness

1.4.1 Rational & Irrational Numbers

Definition 1.11. A real number is *rational* if it can be written in the form $\frac{p}{q}$, where $p,q \in \mathbb{Z}$ and $p \neq 0$. The set of all such numbers is denoted by $\mathbb{Q}$. A real number that is not rational is *irrational*.

Let $a,b \in \mathbb{Q}$ and $c \in \mathbb{R} \setminus \mathbb{Q}$. Then,

- $ab \in \mathbb{Q}$
- $a + b \in \mathbb{Q}$
- $ac \notin \mathbb{Q}$
- $a + c \notin \mathbb{Q}$

Between any two distinct real numbers there is a rational number.

Let $a < b$. There is an infinite number of rational numbers in the open interval (a,b).

Between any two distinct real numbers there is an irrational number.

Let $a < b$. There is an infinite number of irrational numbers in the open interval (a,b).

1.5 Boundedness

Definition 1.12. A **non-empty** set A of real numbers is:

- *bounded above* if $\exists U$ such that $a \leq U$ for all $a \in A$;
- *bounded below* if $\exists L$ such that $a \geq L$ for all $a \in A$;
- *bounded* if it is bounded above **and** below.

Definition 1.13. A real number u is a *least upper bound* of a set A of real numbers if:

- u is an upper bound of A;
- if U is any upper bound of A, then $u \leq U$.

u is also called the *supremum* of A, and is denoted $\sup A$. A real number l is a *greatest lower bound* of a set A of real numbers if:

- l is a lower bound of A;
- if L is any lower bound of A, then $l \geq L$.

l is also called the *infimum* of A, and is denoted $\inf A$.

The supremum and infimum of a set are unique.

1.5.1 Axioms of the Real Numbers

- *Closure under addition*: For $x,y \in \mathbb{R}$, $x + y \in \mathbb{R}$.
- *Associativity of addition*: For $x,y,z \in \mathbb{R}$, $(x + y) + z = x + (y + z)$.
- *Commutativity of addition*: For $x,y \in \mathbb{R}$, $x + y = y + x$.
- *Existence of an additive identity*: There exists a number, 0, such that for all $x \in \mathbb{R}$, $x+0 = 0+x = x$
- *Existence of additive inverses*: For $x \in \mathbb{R}$, thre exists a number $-x$ such that $x+(-x) = (-x)+x = 0$.
- *Closure under multiplication*: For $x,y \in \mathbb{R}$, $xy \in \mathbb{R}$.
- *Associativity of multiplication*: For $x,y,z \in \mathbb{R}$, $(xy)z = x(yz)$.
- *Commutativity of multiplication*: For $x,y \in \mathbb{R}$, $xy = yx$.
- *Existence of an multiplicative identity*: There exists a number, 1, such that for all $x \in \mathbb{R}$, $x \cdot 1 = 1 \cdot x = x$
- *Existence of multiplicative inverses*: For all $x \in \mathbb{R}, x \neq 0$, there exists a number x^{-1} such that $x \cdot x^{-1} = x^{-1} \cdot x = 1$
- *Distribution of multiplication over addition*: For $x,y,z \in \mathbb{R}$, $x(y + z) = xy + xz$.
- *Trichotomy*: For $x,y \in \mathbb{R}$, exactly one of the following statements holds at once: $x < y$, $x = y$, $x > y$.
- *Transitivity*: For $x,y,z \in \mathbb{R}$, if $x < y$ and $y < z$, then $x < z$.
- *Adding to an inequality*: For $x,y,z \in \mathbb{R}$, if $x < y$ then $x + z < y + z$.
- *Multiplying an inequality*: For $x,y,z \in \mathbb{R}$, if $x < y$ and $z > 0$, then $xz < yz$.
- *Non-degeneracy*: $1 \neq 0$
- *Completeness*: Every non-empty subset that is bounded above has a least upper bound.

Now that we have completed MA132, MA138 and MA106, the reals can be more concisely axiomatically defined as the 4-tuple $(\mathbb{R}, +, \cdot, <)$ such that $\mathbb{R}$ is an Dedekind-complete totally-ordered field.

The completeness axiom distinguishes the reals from the rationals.

There are several equivalent statements of the completeness axiom:

- Every non-empty set of real numbers that is bounded above has a least upper bound.
- Every non-empty set of real numbers that is bounded below has a greatest lower bound.
- Every bounded increasing sequence of real numbers is convergent.
- Every bounded decreasing sequence of real numbers is convergent.

- *Bolzano-Weierstrass Theorem*: Every bounded sequence has a convergent subsequence.
- *Intermediate Value Theorem*: Every continuous function that attains both positive and negative values has a root.
- *Cauchy Completeness*: Every Cauchy sequence of real numbers is convergent.
- Every infinite decimal sequence is convergent.

Statements 3 and 4 imply that a monotonic sequence converges if and only if it is bounded.

Definition 1.14. A sequence, (a_n), has the *Cauchy* property if $\forall\, \epsilon > 0\, \exists\, N \in \mathbb{N}$ such that $|a_n - a_m| < \epsilon\, \forall\, n,m > N$.

A sequence of real numbers is Cauchy if and only if it is convergent.

Definition 1.15. A sequence is *strictly contracting* if for some number $0 < l < 1$ (the *contraction factor*), $|a_{n+1} - a_n| \leq l|a_n - a_{n-1}|$ for all n.

A strictly contracting sequence is Cauchy.

Definition 1.16. A positive real number x has a representation as an *infinite decimal* if there is a non-negative integer d_0 and a sequence (d_n) with $d_n \in \{0,1,\cdots,9\}$ for all natural n such that the sequence defined by $\sum_{i=0}^{n} d_i 10^{-i}$ converges to x. In this case, we write $x = d_0.d_1d_2d_3\cdots$. A negative real number x has a representation as the infinite decimal $(-d_0).d_1d_2d_3\cdots$ if $-x$ has a representation as the infinite decimal $d_0.d_1d_2d_3\cdots$.

Every infinite decimal $\pm d_0.d_1d_2d_3...$ represents a real number.

Suppose a positive real number has two different representations as an infinite decimal. Then, one of these is finite, and the other ends with a recurring string of nines.

Definition 1.17. An infinite decimal $\pm d_0.d_1d_2d_3\cdots$ is:

- *terminating* if it ends in repeated zeros;
- *recurring* if there exists $N,r \in \mathbb{N}$ such that $d_n = d_{n+r}$ for all $n > N$;
- *non-recurring* if is neither terminating nor recurring.

Characterisation of terminating decimals: A number x can be represented by a terminating decmial if and only if $x = \frac{p}{q}$, $p,q \in \mathbb{N}$ and the only prime factors of q are 2 and 5.

Every recurring decimal represents a rational number.

Every rational number can be represented by a recurring infinite decimal or a terminating infinite decimal.

Complete Classification:

- Every real number has at least one decimal representation, and every decimal represents a real number.
- The rationals are the set of terminating or recurring deecimals.
- The irrationals are the set of non-recurring decimals.
- If a number has two distinct representations, then one will terminate and the other will end with a recurring string of nines.

1.6 Series

Definition 1.18. A *series* is an expression of the form $\sum_{n=1}^{\infty} a_n$, where a_n is some sequence.

Definition 1.19. The *partial sums* of a series, $\sum_{n=1}^{\infty} a_n$ is a sequence (s_n) given by $s_n = \sum_{i=1}^{n} a_i$.

If s_n is the sequence of partial sums of $\sum_{n=1}^{\infty} a_n$, then,

- $\sum_{n=1}^{\infty} a_n$ converges if (s_n) converges. If $(s_n) \to S$, then $\sum_{n=1}^{\infty} a_n = S$.
- $\sum_{n=1}^{\infty} a_n$ diverges if (s_n) diverges.
- $\sum_{n=1}^{\infty} a_n$ diverges to (minus) infinity if (s_n) diverges to (minus) infinity.

Definition 1.20. A series of the form $\sum_{n=1}^{\infty} x^n$ is called a *geometric* series.

The geometric series $\sum_{n=1}^{\infty} x^n$ is convergent if $|x| < 1$, with the sum being given by $\frac{1}{1-x}$. The series diverges if $|x| \geq 1$.

The series $\sum_{n=1}^{\infty} \frac{1}{n}$ is the *harmonic series.* The harmonic series diverges.

1.6.1 Properties of Convergent Series

Sum Rule for Series: Suppose $\sum_{n=1}^{\infty} a_n$ and $\sum_{n=1}^{\infty} b_n$ are convergent series. Then, for all $c,d \in \mathbb{R}$, $\sum_{n=1}^{\infty}(ca_n + db_n)$ is a convergent series, and is equal to $c\sum_{n=1}^{\infty} a_n + d\sum_{n=1}^{\infty} b_n$.

Shift Rule for Series: Let $N \in \mathbb{N}$. Then, the series $\sum_{n=1}^{\infty} a_n$ converges if and only if the series $\sum_{n=1}^{\infty} a_{N+n}$ converges.

Boundedness Condition: Suppose $a_n \geq 0$ for all n. Then, $\sum_{n=1}^{\infty} a_n$ converges if and only if the sequence of partial sums $(s_n) = \sum_{i=1}^{n} a_i$ is bounded.

Null Sequence Test: If (a_n) is not a null sequence, then $\sum_{n=1}^{\infty} a_n$ diverges.

Comparison Test: Suppose $0 \leq a_n \leq b_n$ for all n. If $\sum_{n=1}^{\infty} b_n$ converges, then $\sum_{n=1}^{\infty} a_n$ converges and $\sum_{n=1}^{\infty} a_n \leq \sum_{n=1}^{\infty} b_n$

Comparison Test: Suppose $0 \leq a_n \leq b_n$ for all n. If $\sum_{n=1}^{\infty} a_n$ diverges, then $\sum_{n=1}^{\infty} b_n$ diverges.

Definition 1.21. $e \coloneqq \sum_{n=0}^{\infty} \frac{1}{n!}$

$e = \lim_{n\to\infty} \left(1 + \frac{1}{n}\right)^n$.

e is irrational.

Ratio Test: Suppose $a_n > 0$ for all n, and $\frac{a_{n+1}}{a_n} \to l$. Then, $\sum_{n=1}^{\infty} a_n$ converges if $0 \leq l < 1$, and diverges if $l > 1$. If $l = 1$, the ratio test is inconclusive.

$$\int_{m}^{n+1} f(x)dx \leq \sum_{k=m}^{n} f(k) \leq \int_{m-1}^{n} f(x)dx$$

Integral Test for Convergence: Suppose the function $f : [1,\infty) \to \mathbb{R}$ is non-negative and decreasing. Then $\sum_{n=1}^{\infty} f(n)$ converges if and only if the increasing sequence $(\int_1^n f(x)dx)$ is bounded, or equivalently, the sequence $(\int_1^n f(x)dx)$ is convergent.

Integral Test for Divergence: Suppose the function $f : [1,\infty) \to \mathbb{R}$ is non-negative and decreasing. Then $\sum_{n=1}^{\infty} f(n)$ diverges if and only if the increasing sequence $(\int_1^n f(x)dx)$ is unbounded, or equivalently, the sequence $(\int_1^n f(x)dx)$ is divergent.

The series $\sum_{n=1}^{\infty} \frac{1}{n^p}$ converges for $p > 1$ and diverges for $0 < p \leq 1$.

Definition 1.22. A series $\sum_{n=1}^{\infty} a_n$ is *absolutely convergent* if the series $\sum_{n=1}^{\infty} |a_n|$ is convergent.

If a series is absolutely convergent, it is convergent.

Definition 1.23. A series $\sum_{n=1}^{\infty} a_n$ is *conditionally convergent* if the series $\sum_{n=1}^{\infty} a_n$ is convergent, but $\sum_{n=1}^{\infty} |a_n|$ is divergent.

1.6.2 Alternating Series

Definition 1.24. A series of the form $\sum_{n=1}^{\infty}(-1)^{n+1}a_n$, where $a_n \geq 0$ for all n is called an *alternating series.*

Alternating Series Test: Suppose (a_n) is decreasing and null. Then, the alternating series $\sum_{n=1}^{\infty}(-1)^{n+1}a_n$ is convergent.

1.6.3 General Series

Ratio Test: Suppose $a_n \neq 0$ for all n and $\left|\frac{a_{n+1}}{a_n}\right| \to l$. Then, $\sum_{n=1}^{\infty} a_n$ converges absolutely (and hence converges) if $0 \leq l < 1$ and diverges if $l > 1$.

Ratio Test: Suppose $a_n \neq 0$ for all n and $\left|\frac{a_{n+1}}{a_n}\right| \to \infty$. Then, $\sum_{n=1}^{\infty} a_n$ diverges.

1.7 Riemann's Rearrangement Theorem

Definition 1.25. The sequence (b_n) is a *rearrangement* of (a_n) if there exists a bijection $\sigma : \mathbb{N} \to \mathbb{N}$ such that $b_n = a_{\sigma(n)}$ for all n.

Suppose $\sum_{n=1}^{\infty} a_n$ is a convergent series, and $a_n \geq 0$ for all n. If (b_n) is a rearrangement of (a_n), then $\sum_{n=1}^{\infty} b_n$ converges and $\sum_{n=1}^{\infty} b_n = \sum_{n=1}^{\infty} a_n$

Suppose $\sum_{n=1}^{\infty} a_n$ is an absolutely convergent series. If (b_n) is a rearrangement of (a_n), then $\sum_{n=1}^{\infty} b_n$ converges and $\sum_{n=1}^{\infty} b_n = \sum_{n=1}^{\infty} a_n$

If a series is conditionally convergent, then the series formed from the subsequence of positive terms diverges to infinity, and the series formed from the subsequence of negative terms diverges to minus infinity.

Riemann's Rearrangement Theorem: Suppose $\sum_{n=1}^{\infty} a_n$ is a conditionally convergent series. Then, for every real number, r, there is a rearrangement (b_n) of (a_n) such that $\sum_{n=1}^{\infty} b_n = r$.

Additionally any conditionally convergent series can be rearranged to diverge to infinity or minus infinity.

2 Analysis II

2.1 Functions

2.1.1 Terminology & Notation

Given two sets, A and B, a *function* from A to B assigns every element in A an element in B. We write $f : A \to B$. If we are talking about elements $a \in A$ and $b \in B$, we write $f : a \mapsto b$.

If every element in A is mapped to a distinct element in B, the function is *injective.* This property can be written $f(a) = f(b) \implies a = b$ or $a \neq b \implies f(a) \neq f(b)$.

If every element in B has an origin element, the function is *surjective.* This property can be written as $\forall b \in B, \exists a \in A$ such that $f(a) = b$.

If a function is both injective and surjective, it is *bijective.*

For this function, A is the *domain* of f, and B is the *codomain.*

The set $\{f(x) : x \in A\}$ is the *image* of the function, and is a subset of the codomain.

Do not use the term "*range*" if possible, as it can refer to both the codomain and the image.

An *interval* of the real line is a subset, I, of $\mathbb{R}$ with the property that if $x < y < z$ and $x \in I$ and $z \in I$, then $y \in I$. That is to say, an interval contains all points between its endpoints, but the endpoints may or may not be included themselves.

$\{x : a \leq x \leq b\} = [a,b]$	a *closed* interval
$\{x : a < x < b\} = (a,b)$	an *open* interval
$\{x : a \leq x < b\} = [a,b)$	a *half-open* interval
$\{x : a \leq x\} = [a,\infty)$	a *half-infinite* interval

Note that we use an open interval bracket whenever we have ∞ as one of the endpoints.

Two functions, $f : A \to B$ and $g : C \to D$ are *equal* if $A = C$, $B = D$ and $f(x) = g(x)$ for all $x \in A$.

2.1.2 Continuity

Definition 2.1. *Continuity*: A function $f : I \to R$ is said to be continuous at $c \in I$ if $\forall \epsilon > 0, \exists \delta > 0$, such that if $x \in I$ and $|x - c| < \delta$, then $|f(x) - f(c)| < \epsilon$.

Let $L^+ = \lim_{x \to c^+} f(x)$ and $L^- = \lim_{x \to c^-} f(x)$. Using these quantities, we define the three classes of discontinuities:

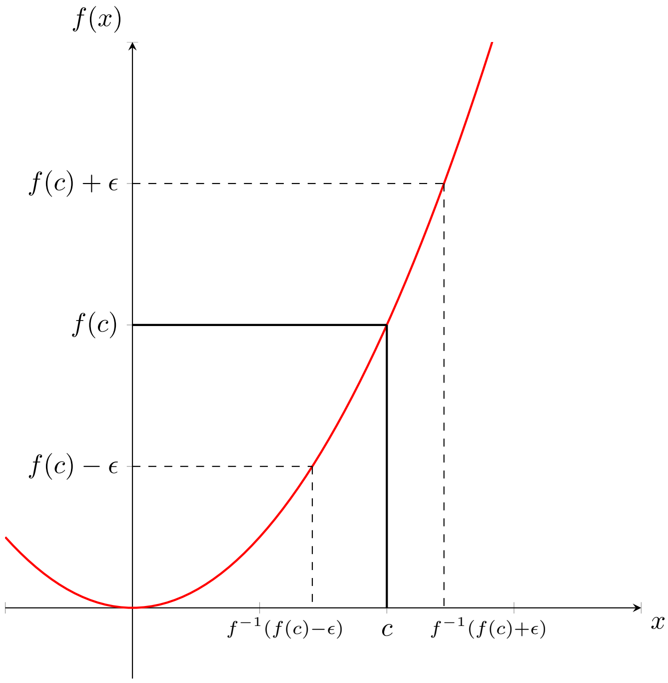
- A *removable discontinuity* is where $L^+ = L^- = L$, $f(c)$ exists, and $f(c) \neq L$. If $L^+ = L^-$ but $f(c)$ is undefined, then $f(c)$ is instead a *removable singularity.*
- A *jump discontinuity* is where $L^+ \neq L^-$. $f(c)$ can take any value.
- A *essential discontinuity* is where at least one of L^+ and L^- do not exist.

Example. The function $x \mapsto x$ is continuous for all x.

Proof. Let $\delta = \epsilon$, so if $|x - c| < \delta$, then $|x - c| < \epsilon$ and $|x - c| = |f(x) - f(c)| < \epsilon$. ■

Example. The function $x \mapsto x^2$ is continuous for all x.

Proof. We present a graphical method for working through these types of questions when the function is easy to plot.



So clearly, if x is less than $\min\left(|c - f^{-1}(f(c) - \epsilon|), |c - f^{-1}(f(c) + \epsilon)|\right)$ away from c, then $|f(x) - f(c)| < \epsilon$. Thus, if we let $\delta = \min\left(\left|c - \sqrt{c^2 - \epsilon}\right|, \left|c - \sqrt{c^2 + \epsilon}\right|\right)$, then $|f(x) - f(c)| < \epsilon$ as required. ■

Sequential Continuity: Let $f : I \to \mathbb{R}$ and $c \in I$. Then, f is continuous at c if and only if, for every sequence (x_n) of points in I which converges to c, we have $f(x_n) \to f(c)$ as $n \to \infty$.

Algebra of Continuous Functions: Let $f,g : I \to \mathbb{R}$ be functions continuous at $c \in I$. Then,

- $f + g$ is continuous at c;
- fg is continuous at c;
- $\frac{f}{g}$ and is continuous at c if $g(c) \neq 0$.

Continuity of Polynomials and Rational Functions: If p is a polynomial, then p is continuous over $\mathbb{R}$. If $r = \frac{p}{q}$ is the ratio of two polynomials, it is continuous over $\mathbb{R}$ wherever $q \neq 0$.

Proof. $f(x_n) \to f(c)$ and $g(x_n) \to g(c)$, so applying algebra of sequences, we have $(f + g)(x_n) = f(x_n) + g(x_n) \to f(c) + g(c)$, so $(f + g)(x)$ is continuous by sequential continuity.

The proofs for the continuity of the product and ratio of continuous functions are similar. ■

Composition of Continuous Functions: Let $f : I \to \mathbb{R}$ and $g : X \to I$. If g is continuous at c and f is continuous at $g(c)$, then the composition $f \circ g$ is continuous at c.

Proof. Let (x_n) be a sequence in X converging to c. Then, $g(x_n) \to g(c) \in I$ and hence $f(g(x_n)) \to f(g(c))$. ■

2.2 The Intermediate Value Theorem

Intermediate Value Theorem: Let $f : [a,b] \to \mathbb{R}$ be continuous and suppose $f(a) < k < f(b)$. Then, there exists $c \in [a,b]$ such that $f(c) = k$

Proof. Let $f(a) < u < f(b)$ and let $S = \{x \in [a,b] : f(x) \leq u\}$. S is non-empty as $a \in S$ and is bounded above by b. By completeness, the supremum, $c = \sup S$ exists. We will show that $f(c) = u$.

Let $\epsilon > 0$. As f is continuous, there exists $\delta > 0$ such that $|f(x) - f(c)| < \epsilon$ whenever $|x - c| < \delta$, so we have;

$$f(x) - \epsilon < f(c) < f(x) + \epsilon \qquad (\star)$$

for all $x \in (c - \delta, c + \delta)$ by the interval property. By the properties of the supremum, there exists some $a^* \in (c - \delta, c]$ that is contained in S. By construction, a^* is within δ of c, so $|a^* - c| < \delta$ holds, and we may use the RHS of $(\star)$ to write:

$$f(c) < f(a^*) + \epsilon \leq u + \epsilon$$

Picking $a^{**} \in (c, c + \delta)$, we know $a^{**} \notin S$ since c is the supremum of S and $a^{**} > c$ by construction. Again, a^{**} is within δ of c, so,

$$f(c) > f(a^{**}) - \epsilon > u - \epsilon$$

and combining both inequalities, we have,

$$u - \epsilon < f(c) < u + \epsilon$$

and u is the only value of $f(c)$ such that the above inequality holds for all $\epsilon > 0$, so $f(c) = u$. ■

Existence of Square Roots: Every positive real number has a unique positive square root.

Proof. Let r be a positive real number and consider $f : x \mapsto x^2$. $f(0) < r$ and $f(r+1) = r^2 + 2r + 1 > r$, so there exists a number $c \in [0, r+1]$ such that $f(c) = r$. It follows that $c^2 = R$ and c is a positive square root of r.

Now suppose two distinct numbers positive real numbers, x,y exist such that $f(x) = r$ and $f(y) = r$. By trichotomy, and without loss of generality, suppose $x < y$. As x,y are positive and real, we have $x^2 < y^2$, contradicting that $f(x) = r$ and $f(y) = r$. It follows that the positive square root of any positive real number is unique. ■

Continuous Image of an Interval: If $f : I \to \mathbb{R}$ is continuous over I, then the image of f is also an interval.

Proof. If x and y are in the image of f, then by the IVT, every point between x and y is also in the image of f, so the image of f is an interval. ■

Existence of Inverses: Let $f : [a,b] \to \mathbb{R}$ be continuous and strictly increasing. Then f has a continuous inverse, f^{-1} defined over its image.

Proof. Let $f(a) = c$ and $f(b) = d$. Since f is increasing, the image of f lies between c and d. In fact, the image of f is exactly $[c,d]$.

For each y, there is a unique number x such that $f(x) = y$ as f is strictly increasing, so we let $f^{-1} = g : y \mapsto x$. By construction, g is increasing.

Let $\epsilon > 0$, and suppose $f(x) = y \in (c,d)$, so $f(x - \epsilon) < y < f(x + \epsilon)$, so there exists δ such that,

$$f(x - \epsilon) < y - \delta < y < y + \delta < f(x + \epsilon)$$

and for any $z \in (y-\delta,y+\delta)$, we have

$$x-\epsilon < g(z) < x+\epsilon$$

so $|g(z)-g(y)| < \epsilon$.

If instead $y=a$ or $y=b$, the argument is the same, but with $f(x-\epsilon)$ and $f(x+\epsilon)$ replaced by c or d. ■

Existence of Roots For each positive real x and natural n, there is a unique positive nth root $x^{\frac{1}{n}}$, and the map $x \mapsto x^{\frac{1}{n}}$ is continuous.

2.3 The Extreme Value Theorem

Extreme Value Theorem: Let $f : [a,b] \to \mathbb{R}$ be continuous over $[a,b]$. Then, there exists numbers $c,d \in [a,b]$ such that $f(c) \le f(x) \le f(d)$ for all $x \in [a,b]$.

Proof. Let M be the least upper bound of the set $S = \{f(x) : x \in [a,b]\}$. If there is no point, c, in the interval where $f(c) = M$, then, the function $g(x) = M - f(x)$ is strictly positive and continuous over $[a,b]$. By the algebra of limits, $\frac{1}{g(x)}$ is continuous, and therefore bounded. Let R be an upper bound of $1/g(x)$ over the interval $[a,b]$, noting that $R > 0$. Then, $\frac{1}{R} \le g(x) = M - f(x)$, so $f(x) \le M - \frac{1}{R}$, and $M - \frac{1}{R}$ is an upper bound of S. As R is positive, $M - \frac{1}{R} < M$, contradicting that M is the least upper bound of S.

It follows that the point c must exist, and f attains the maximum, $f(c)$ over $[a,b]$. The proof for the minimum is similar. ■

2.4 Boundedness

Definition 2.2. A function, $f : I \to \mathbb{R}$, is:

- *bounded above* if $\exists M \in \mathbb{R}$ such that $f(x) \le M \, \forall \, x \in I$. M is then an *upper bound* of the sequence;
- *bounded above* if $\exists m \in \mathbb{R}$ such that $f(x) \ge m \, \forall \, x \in I$. m is then a *lower bound* of the sequence;
- *bounded* if it is bounded above **and** below.

Boundedness of Continuous Functions: If $f : [a,b] \to \mathbb{R}$ is continuous, then f is bounded.

Proof. Suppose f is continuous, but unbounded. We construct a sequence, (x_n), where $|f(x_n)| \ge n$ for all n. By Bolzano-Weierstrass, we can construct a subsequence, (x_{n_i}) which converges to some value, say, x. Since the interval over which f is defined is closed, $x \in [a,b]$ by the closed interval rule. By sequential continuity, $f(x_{n_i}) \to f(x)$, but this is impossible as $f(x_{n_i})$ become arbitrarily large by the construction of (x_n). ■

2.5 Power Series

Definition 2.3. A *power series* is a series of the form $\sum_{n=0}^{\infty} a_n(x-c)^n$.

We mostly focus on the specific case $c = 0$, with many theorems easily transferring over to the general case.

Radius of Convergence I: Let $\sum_{n=0}^{\infty} a_n x^n$ be a power series with $\sum_{n=0}^{\infty} a_n t^n$ convergent. Then, $\sum_{n=0}^{\infty} a_n x^n$ converges absolutely for all x such that $|x| < |t|$.

Proof. Since $\sum_{n=0}^{\infty} a_n t^n$ converges, we know that $a_n t^n \to 0$ as $n \to \infty$ so the sequence of partial sums is bounded by some M such that $|a_n t^n| < M$ for all n. Now,

$$\begin{aligned}\sum_{n=0}^{N} |a_n x^n| &= \sum_{n=0}^{N} |a_n t^n| \left|\frac{x}{t}\right|^n \\ &\leq M \sum_{n=0}^{N} \left|\frac{x}{t}\right|^n \\ &\leq M \sum_{n=0}^{\infty} \left|\frac{x}{t}\right|^n \\ &= M \frac{1}{1 - \left|\frac{x}{t}\right|}\end{aligned}$$

which is finite, so the series converges absolutely. ■

Radius of Convergence II: Let $\sum_{n=0}^{\infty} a_n x^n$ be a power series. Then, one of the following statements holds:

- There is a positive real R such that the series converges if $|x| < R$ and diverges if $|x| > R$. If such a number exists, it is called the *radius of convergence.*
- The series converges only if $x = 0$. In this case, we say the radius of convergence is 0.
- The series converges for all real x. In this case, we say the radius of convergence is ∞.

Absolute Series: Let $\sum_{n=0}^{\infty} a_n x^n$ be a power series with radius of convergence R. Then, $\sum_{n=0}^{\infty} |a_n| x^n$ also has radius of convergence R.

Geometric Series I: The series $\sum_{n=0}^{\infty} x^n$ has radius of convergence $R = 1$.

Geometric Series II: If p is real, the series $\sum_{n=0}^{\infty} p^n x^n$ has radius of convergence $R = \frac{1}{|p|}$.

Proof. $\sum_{n=0}^{\infty} p^n x^n = \sum_{n=0}^{\infty} (px)^n$, which converges if $|px| \leq 1$, so $|x| \leq \frac{1}{|p|}$. ■

The Log Series: The series $\sum_{n=0}^{\infty} \frac{x^n}{n}$ has radius of convergence $R = 1$.

Continuity of Convergent Power Series: Let $\sum_{n=0}^{\infty} a_n x^n$ be a power series with radius of convergence R. Then, the function, $x \mapsto \sum_{n=0}^{\infty} a_n x^n$ is continuous over $(-R,R)$.

Proof. Let $x \in (-R,R)$, and T such that $|x| < T < R$. It follows that $T \in (-R,R)$ so $\sum_{n=0}^{\infty} |a_n| T^n$ converges, so for each $\epsilon > 0$ there exists N for which,

$$\sum_{n=N+1}^{\infty} |a_n| T^n < \frac{\epsilon}{3}$$

Now, if $|y - x| < T - |x|$, $|y| < T$ and $x < T$, so,

$$\sum_{n=N+1}^{\infty} |a_n||x|^n < \frac{\epsilon}{3} \text{ and } \sum_{n=N+1}^{\infty} |a_n||y|^n < \frac{\epsilon}{3}$$

The partial sum $\sum_{n=0}^{N} |a_n|y^n$ is a polynomial in y, and polynomials are continuous, so there exists some δ_0 such that if $|y-x| < \delta_0$,

$$\left|\sum_{n=0}^{N} a_n y^n - \sum_{n=0}^{N} a_n x^n\right| < \epsilon/3$$

So, letting $\delta = \min(\delta_0, T - |x|)$, if $|y-x| < \delta$, we have,

$$\begin{aligned}\left|\sum_{n=0}^{\infty} a_n y^n - \sum_{n=0}^{\infty} a_n x^n\right| &\leq \left|\sum_{n=N+1}^{\infty} a_n y^n\right| + \left|\sum_{n=0}^{N} a_n y^n - \sum_{n=0}^{N} a_n x^n\right| + \left|\sum_{n=N+1}^{\infty} a_n x^n\right| \\ &\leq \sum_{n=N+1}^{\infty} |a_n||y|^n + \left|\sum_{n=0}^{N} a_n y^n - \sum_{n=0}^{N} a_n x^n\right| + \sum_{n=N+1}^{\infty} |a_n||x|^n < \epsilon\end{aligned}$$

■

2.5.1 The Exponential Function

The *exponential series* $\exp(x) = \sum_{n=0}^{\infty} \frac{x^n}{n!}$ converges.

Proof.

$$\begin{aligned}\frac{x^{n+1}}{(n+1)!}\frac{n!}{x^n} &= \frac{x}{n+1} \\ &\to 0\end{aligned}$$

so the series converges by the ratio test. ■

The function $x \mapsto \exp(x)$ is continuous by the continuity of convergent power series.

The Characteristic Property of the Exponential: $\exp(x+y) = \exp(x)\exp(y)$. $\exp(1) = e$.

Proof. For a fixed $z \in \mathbb{R}$, consider the function $f(x) = \exp(x)\exp(z-x)$. Differentating with respct to x, we have $f'(x) = \exp(x)\exp(z-x) - \exp(x)\exp(z-x) = 0$, so $f(x)$ must be a constant function by the MVT. At $x = 0$, $f(x) = \exp(z)$, but, as $f(x)$ is constant, we must have $f(x) = \exp(z)$ for all x, so $\exp(x)\exp(z-x) = \exp(z)$ for all x. Let $z = x + y$, and we have $\exp(x)\exp(y) = \exp(x+y)$. ■

Inequalities for the Exponential:

- $1 + x \leq e^x$
- $e^x \leq \frac{1}{1-x}$ if $x < 1$.

Proof. If $x \geq 0$ then $e^x = 1 + x + \frac{x^2}{2} + \cdots \geq 1 + x$, and if $0 \leq x < 1$, $e^x = 1 + x + \frac{x^2}{2} + \frac{x^3}{6} + \cdots \leq 1 + x + x^2 + x^3 + \cdots = \frac{1}{1-x}$.

If $x < 0$, let $u = -x$ so $e^u \geq 1 + u$ implies $e^{-x} \geq 1 - x \Leftrightarrow \frac{1}{e^x} \geq 1 - x \Leftrightarrow \frac{1}{1-x} \geq e^x$, so the second inequality holds for all $x < 1$.

If $x \leq -1$, then $1 + x \leq 0$, but $e^x > 0$, so $e^x \geq 1 + x$ for $x \leq -1$. Now, if $-1 < x < 0$, then $0 < u < 1$ and so $e^u \leq \frac{1}{1-u}$, so $e^{-x} \leq \frac{1}{1+x} \Leftrightarrow \frac{1}{e^x} \leq \frac{1}{1+x} \Leftrightarrow \frac{1}{+}x \leq e^x$ ■

The exponential function is strictly increasing, and its image is $(0,\infty)$.

Proof. Suppose $x < y$. Then, $e^y = e^{y-x}e^x \geq (1 + y - x)e^x > e^x$.

Since $e^x \geq 1 + x$, the exponential function takes arbitrarily large values for large choices of x, and since $e^{-x} = \frac{1}{e^x}$, the exponential function takes arbitrarily small values for very large negative choices of x. By the IVT, the exponential takes all positive values. ■

2.5.2 The Logarithmic Function

The exponential function maps R to $(0,\infty)$ and is continuous and strictly increasing. So, by the IVT, the exponential function has a continuous inverse, called the *natural logarithm*, defined over $(0,\infty)$.

The function $\log : (0,\infty) \mapsto \mathbb{R}$ satisfies $e^{\log x} = x$ for all positive real x, and $\log(e^y) = y$ for all real y. For all positive real a,b, we have $\log(ab) = \log a + \log b$.

Powers: If $x > 0$ and $p \in \mathbb{R}$, we define $x^p = \exp(p \log x)$

Tangent to the Logarithm: If $x > 0$, then $\log x \leq x - 1$.

2.6 Limits

Limits of Functions: Let I be an open interval and f a real-valued function defined over I, except possibly at a point $c \in I$. We write,

$$\lim_{x \to c} f(x) = L$$

if for every $\epsilon > 0$, there exists $\delta > 0$ such that if $0 < |x - c| < \delta$, then $|f(x) - L| < \epsilon$.

Limits and Continuity: If $f : I \to \mathbb{R}$ is defined over the open interval I and $c \in I$, then f is continuous at c if and only if $\lim_{x \to c} f(x) = f(c)$.

Continuous and Sequential Limits: If $f : I \backslash \{c\} \to \mathbb{R}$ is defined over the interval $I \backslash \{c\}$, then $\lim_{x \to c} f(x) = L$ if and only if for every sequence (x_n) of points in $I \setminus \{c\}$ which converges to c, we have $f(x_n) \to L$.

Algebra of Limits: If f,$g : I \backslash \{c\} \to \mathbb{R}$ are defined over the interval $I \backslash \{c\}$ and $\lim_{x \to c} f(x)$ and $\lim_{x \to c} g(x)$ exist, then,

- $\lim_{x \to c}(f(x) + g(x)) = \lim_{x \to c} f(x) + \lim_{x \to c} g(x)$
- $\lim_{x \to c} f(x)g(x) = \lim_{x \to c} f(x) \lim_{x \to c} g(x)$
- $\lim_{x \to c} \frac{f(x)}{g(x)} = \frac{\lim_{x \to c} f(x)}{\lim_{x \to c} g(x)}$ if $\lim_{x \to c} g(x) \neq 0$.

One Sided Limits: Let $f : [a,b] \to \mathbb{R}$. We write,

$$\lim_{x \to c^+} f(x) = L$$

if for every $\epsilon > 0$, there exists $\delta > 0$ such that if $c < x < c + \delta$, then $|f(x) - L| < \epsilon$. The limit,

$$\lim_{x \to c^-} f(x) = L$$

is defined similarly.

Infinite Limits: Let I be an open interval and f a real-valued function defined over I, except possibly at a point $c \in I$. We write,

$$\lim_{x \to c} f(x) = \infty$$

if for every $M > 0$ there is a $\delta > 0$ such that if $0 < |x - c| < \delta$ then $f(x) > M$. The limit,

$$\lim_{x \to c} f(x) = -\infty$$

is defined similarly.

Limits at Infinity: If $f : \mathbb{R} \to \mathbb{R}$, we write,

$$\lim_{x \to \infty} f(x) = L$$

if for every $\epsilon > 0$ there is an N such that if $x > N$ then $|f(x) - L| < \epsilon$.

Uniqueness of Limits: If $f(x) \to M$ as $x \to c$, and $f(x) \to L$ as $x \to c$, then $M = L$.

Sandwich Theorem for Limits: Let I be an interval containing the point a. Let g,f,h be functions defined over I, except possibly at a. If for every $x \in I$, we have $g(x) \leq f(x) \leq h(x)$ and $\lim_{x\to a} g(x) = \lim_{x\to a} h(x) = L$, then $\lim_{x\to a} f(x) = L$.

Note that a does not have to lie within the interior of I, and can be an endpoint, with the limits above being evaluated as one-sided limits. Similarly, the statement holds for infinite intervals, where $x \to \pm\infty$.

2.7 The Derivative

Let $f : I \to \mathbb{R}$ and $c \in I$. f is *differentiable* at c if,

$$\lim_{h \to 0} \frac{f(c+h) - f(c)}{h}$$

exists. If so, we denote this limit $f'(c)$.

Letting $x = c + h$, we can equivalently write the derivative as,

$$\lim_{x \to c} \frac{f(x) - f(c)}{x - c}$$

Differentiability and Continuity: If $f : I \to \mathbb{R}$ is differentiable at $c \in I$ then f is continuous at c.

Proof. $f(x) - f(c) = \frac{f(x)-f(c)}{x-c} \cdot (x - c)$. $\frac{f(x)-f(c)}{x-c} \to f'(c)$ as $x \to c$ by the definition of a derivative, and $(x - c) \to 0$ as $x \to c$, so $\frac{f(x)-f(c)}{x-c} \cdot (x - c) \to f'(c) \cdot 0 = 0$, so $f(x) - f(c) \to 0$, and $f(x) \to f(c)$. ■

Sum and Product Rules: Suppose f,$g : I \to \mathbb{R}$ are differentiable at $c \in I$. Then, $(f + g)$ and fg are differentiable at c, and,

$$(f+g)'(c) = f'(c) + g'(c)$$
$$(fg)'(c) = f(c)g'(c) + f'(c)g(c)$$

Derivatives of Monomials: If n is a natural, then the derivative of $f : x \mapsto x^n$ is $f' : x \mapsto nx^{n-1}$.

Proof. For $n = 1$, we have $f : x \mapsto x$. For every c, and $h \neq 0$,

$$\lim_{h \to 0} \frac{f(c+h) - f(c)}{h} = \lim_{h \to 0} \frac{c+h-c}{h} = 1$$

Suppose the statement holds for arbitrary fixed $n \geq 1$. Then, $x^{n+1} = xf(x)$, so by the product rule, the derivative is $1 \cdot f(x) + xf'(x) = x^n + xnx^{n-1} = (n+1)x^n$, which is the statment for $n + 1$, completing the inductive step. ■

Chain Rule: Suppose $f : I \to \mathbb{R}$, $g : X \to I$, g is differentiable at $c \in X$ and f is differentiable at $g(c) \in I$. Then, the composition $f \circ g$ is differentiable at c, and,

$$(f \circ g)'(c) = f'(g(c)) \cdot g'(c)$$

2.8 The Mean Value Theorem

Rolle's Theorem: Suppose $f : [a,b] \to \mathbb{R}$ is continuous over $[a,b]$ and differentiable over (a,b), and that $f(a) = f(b)$. Then, there is a point $c \in (a,b)$ such that $f'(c) = 0$.

Proof. If f is constant over the interval, then f' is 0 everywhere over the interval. If not, it takes values distinct from $f(a) = f(b)$.

As f is continuous, there f attains a maximum and minimum within the interval $[a,b]$ by the extreme value theorem. Suppose f attains a maximum at $c \neq a,b$, so $c \in (a,b)$.

If $x \neq c$, then $f(x) \leq f(c)$, as c is a maximum, so

$$f(x) - f(c) \leq 0 \tag{$\star$}$$

Suppose $x > c$. Then, $x - c > 0$, so $\frac{1}{x-c} > 0$. Multiplying both sides of $(\star)$ by $\frac{1}{x-c}$, we have,

$$\frac{f(x) - f(c)}{x - c} \leq 0$$

so $f'(c) \leq 0$

Now suppose $x < c$. Then, $x - c < 0$, so $\frac{1}{x-c} < 0$. Multiplying both sides of $(\star)$ by $\frac{1}{x-c}$, we now need to swap the direction of inequality as we are multiplying by a negative value, so we have,

$$\frac{f(x) - f(c)}{x - c} \geq 0$$

so $f'(c) \geq 0$.

Using the two above equations, we deduce that $f'(c) = 0$ ■

Mean Value Theorem: Suppose $f : [a,b] \to \mathbb{R}$ is continuous over $[a,b]$ and differentiable over (a,b). Then, there is a point $c \in (a,b)$ such that,

$$f'(c) = \frac{f(b) - f(a)}{b - a}$$

This theorem may be more useful in the form,

$$f(b) - f(a) = (b - a)f'(c)$$

Informally, the theorem states that, for any arc between two endpoints, a and b, there exists at least one point between the two endpoints such that the tangent to the arc is parallel to the line that passes through its endpoints.

Another statement of the theorem that more accurately reflects its name, is that there exists at least one point, c, between the two endpoints of the graph of $y = f(x)$ such that the slope at $f(c)$ is equal to the the average slope of the graph between the endpoints.

Proof. Let

$$g(x) = f(x) - xr$$

where $r = \frac{f(b)-f(a)}{b-a}$, which is a constant. Then,

$$\begin{aligned} g(b) - g(a) &= (f(b) - br) - (f(a) - ar) \\ &= f(b) - f(a) - (b - a)r \end{aligned}$$

$$= f(b) - f(a) - (f(b) - f(a))$$
$$= 0$$

So, by Rolle's Theorem, there exists a point $c \in (a,b)$ such that $g'(c) = 0$. So, we have,

$$\begin{aligned} g'(x) &= f'(x) - r \\ g'(c) &= f'(c) - r \\ &= 0 \\ \implies f'(c) &= r \\ &= \frac{f(b) - f(a)}{b - a} \end{aligned}$$

■

Functions with Positive Derivative: If $f : I \to \mathbb{R}$ is differentiable over the open interval I, and $f'(x) > 0$ for all $x \in I$, then f is strictly increasing over I.

Proof. If there were two points a and b such that $a < b$ but $f(a) \geq f(b)$, then $b - a > 0$ and $f(a) - f(b) \geq 0$, so $f'(c) = \frac{f(b)-f(a)}{b-a} \leq 0$ for some point $c \in I$, contradicting that $f'(x) > 0$ for all $x \in I$. ■

Functions with Zero Derivative: If $f : I \to \mathbb{R}$ is differentiable over the open interval I and $f'(x) = 0$ for all $x \in I$, then f is constant over I.

Proof. By the MVT, there exists $c \in I$ such that $(x - a)f'(c) = f(x) - f(a)$. But, $f'(c) = 0$ for all $c \in I$, so $f(x) - f(a) = 0$ and $f(x) = f(a)$. As the choice of x and a were arbitrary, letting $f(a) = k$, we have $f(x) = k$ for all $x \in I$. ■

Extrema and Derivatives: Suppose $f : [a,b] \to \mathbb{R}$ is continuous and is differentiable over (a,b). Then, f attains its maximum and minimum at points within the open interval where $f' = 0$, or at one of the endpoints, a or b.

Example. Find the maximum of xe^{-x} over $\mathbb{R}$.

The derivative of $f(x) = xe^{-x}$ is $f'(x) = (1 - x)e^{-x}$. e^{-x} is positive for all x, and $(1 - x)$ is positive if $x < 1$ and negative if $x > 1$, so the whole expression is positive if $x < 1$ and negative if $x > 1$. By the MVT, the function increases until $x = 1$ and then decreases, so the maximum is attained at $x = 1$, where $f(1) = e^{-}1$.

2.9 Inverses

Derivatives of Inverses: Let $f : (a,b) \to \mathbb{R}$ be differentiable with positive derivative. Then, $g = f^{-1}$ is differentiable, and $g'(x) = \frac{1}{f'(g(x))}$.

Proof. Since f has positive derivative, it is continuous and strictly increasing. Therefore, it has a continuous inverse. Let (c,d) be the range of the image of f, and let $x \in (c,d)$ and $g(x) = y$. Consider

$$g'(x) = \lim_{u \to x} \frac{g(u) - g(x)}{u - x}$$

Let $v = g(u)$. As $u \to x$, we have $g(u) \to g(x)$ as g is continuous at x, so $v \to y$, and,

$$= \lim_{v \to y} \frac{v - y}{f(v) - f(y)}$$

$$= \frac{1}{f'(y)}$$
$$= \frac{1}{f'(g(x))}$$

■

2.10 Power Series II

Differentiability of Power Series I: Let $\sum_{n=0}^{\infty} a_n x^n$ be a power series with radius of convergence R. Then, the series $\sum_{n=0}^{\infty} n a_n x^{n-1}$ has the same radius of convergence.

Proof. The series $\sum_{n=0}^{\infty} |a_n| x^n$ has the same radius of convergence by absolute series theorem. Let $0 < x < y < R$, so $\sum_{n=0}^{\infty} |a_n| x^n$ and $\sum_{n=0}^{\infty} |a_n| y^n$ both converge. It follows that their difference,

$$\sum_{n=0}^{\infty} |a_n| y^n - \sum_{n=0}^{\infty} |a_n| x^n = \sum_{n=0}^{\infty} |a_n|(y^n - x^n)$$

also converges, and thus

$$\sum_{n=0}^{\infty} |a_n| \frac{(y^n - x^n)}{y - x}$$

also converges. But

$$\sum_{n=0}^{\infty} |a_n| \frac{(y^n - x^n)}{y - x} = \sum_{n=1}^{\infty} |a_n|(y^{n-1} + y^{n-2}x + \cdots + x^{n-1})$$

Noting that $y > x$, we have,

$$\geq \sum_{n=1}^{\infty} |a_n|(x^{n-1} + x^{n-2}x + \cdots + x^{n-1})$$
$$\geq \sum_{n=1}^{\infty} |a_n|(\underbrace{x^{n-1} + x^{n-1} + \cdots + x^{n-1}}_{n})$$
$$\geq \sum_{n=1}^{\infty} |a_n| n x^{n-1}$$

so $\sum_{n=0}^{\infty} n|a_n| x^{n-1}$ also converges, further implying that $\sum_{n=0}^{\infty} n a_n x^{n-1}$ converges (absolutely). ■

Differentiability of Power Series II: Let $\sum_{n=0}^{\infty} a_n x^n$ be a power series with radius of convergence R. Then, the function, $x \mapsto \sum_{n=0}^{\infty} a_n x^n$ is continuous and differentiable over $(-R,R)$.

Proof. Let $x \in (-R,R)$, and T such that $|x| < T < R$. It follows that $T \in (-R,R)$ so, by the theorem above, $\sum_{n=0}^{\infty} n|a_n| T^{n-1}$ converges, so for each $\epsilon > 0$ there exists N for which,

$$\sum_{n=N+1}^{\infty} n|a_n| T^{n-1} < \frac{\epsilon}{3}$$

Now, if $|y - x| < T - |x|$, $|y| < T$ and $x < T$, so,

$$\left| \sum_{n=N+1}^{\infty} n a_n x^{n-1} \right| \leq \sum_{n=N+1}^{\infty} n|a_n||x|^{n-1} < \frac{\epsilon}{3}$$

and also

$$\begin{aligned}\left|\sum_{n=N+1}^{\infty} a_n \frac{y^n - x^n}{y - x}\right| &= \left|\sum_{n=N+1}^{\infty} a_n (y^{n-1} + y^{n-2}x + \cdots + x^{n-1})\right| \\ &\le \sum_{n=N+1}^{\infty} |a_n|(|y|^{n+1} + \cdots + |x|^{n-1}) \\ &\le \sum_{n=N+1}^{\infty} n|a_n|T^{n-1} \\ &< \frac{\epsilon}{3}\end{aligned}$$

The finite sum,

$$\sum_{n=1}^{N} a_n(y^{n-1} + y^{n-2}x + \cdots + x^{n-1})$$

is a polynomial in y equal to $\sum_{n=1}^{N} n a_n x^{n-1}$ when $y = x$, so there exists a $delta_0 > 0$ such that if $0 < |y - x| < \delta_0$,

$$\begin{aligned}\left|\sum_{n=1}^{N} a_n \frac{y^n - x^n}{y - x} - \sum_{n=1}^{N} n a_n x^{n-1}\right| &= \left|\sum_{n=1}^{N} a_n (y^{n-1} + y^{n-2}x) + \cdots + x^{n-1} - \sum_{n=1}^{N} n a_n x^{n-1}\right| \\ &< \frac{\epsilon}{3}\end{aligned}$$

So, letting $\delta = \min(\delta_0, T - |x|)$, if $|y - x| < \delta$, we have,

$$\left|\sum_{n=1}^{\infty} a_n \frac{y^n - x^n}{y - x} - \sum_{n=1}^{\infty} n a_n x^{n-1}\right| \le$$

$$\left|\sum_{n=N+1}^{\infty} a_n \frac{y^n - x^n}{y - x}\right| + \left|\sum_{n=1}^{N} a_n \frac{y^n - x^n}{y - x} - \sum_{n=1}^{N} n a_n x^{n-1}\right| + \left|\sum_{n=N+1}^{\infty} n a_n x^{n-1}\right| < \epsilon$$

■

The derivative of the exponential function is equal to the exponential function.

2.11 The Trigonometric Functions

Definition 2.4.

$$\begin{aligned}\cos(x) &:= \sum_{n=0}^{\infty} \frac{(-1)^n}{(2n)!} x^{2n} \\ &= 1 - \frac{x^2}{2!} + \frac{x^4}{4!} - \cdots \\ \sin(x) &:= \sum_{n=0}^{\infty} \frac{(-1)^n}{(2n+1)!} x^{2n+1} \\ &= x - \frac{x^3}{3!} + \frac{x^5}{5!} - \cdots\end{aligned}$$

Addition Formulae: For all $x, y \in \mathbb{R}$,

$$\cos(x \pm y) = \cos(x)\cos(y) \mp \sin(x)\sin(y)$$
$$\sin(x \pm y) = \sin(x)\cos(y) \pm \cos(x)\sin(y)$$

Proof. Let $f(x) = \cos(x)\cos(z-x) - \sin(x)\sin(z-x)$. $f'(x) = 0$, so $f(x)$ is constant by the MVT. When $x = 0$, $f(0) = \cos(0)\cos(z) - \sin(0)\sin(z) = \cos(z)$, so $f(x) = \cos(z)$ for all x. Letting $z = x + y$, we have $\cos(x)\cos(y)\sin(x)\sin(y) = f(x) = \cos(z) = \cos(x + y)$.

The proof for $\sin(x + y)$ is similar. ■

The Circular Property: For all $x \in \mathbb{R}$,

$$\cos^2(x) + \sin^2(y) = 1$$

Proof. In the addition formula for $cos(x + y)$, let $y = -x$. By the even and odd properties of $\cos(x)$ and $\sin(x)$, we have, $1 = \cos(0) = \cos(x - x) = \cos(x)\cos(-x) - \sin(x)\sin(-x) = \cos^2(x) + \sin^2(x)$. ■

2.12 Taylor's Theorem

Cauchy's Mean Value Theorem: If $f{,}g : [a{,}b] \to \mathbb{R}$ are continuous and differentiable over $(a{,}b)$, and $g'(t) \neq 0$ for $t \in (a{,}b)$, then there exists a point c such that,

$$\frac{f'(c)}{g'(c)} = \frac{f(b) - f(a)}{g(b) - g(a)}$$

Proof. Consider $h(x) = f(x)[g(b) - g(a)] - g(x)[f(b) - f(a)]$.

$$\begin{aligned} h(a) &= f(a)g(b) - f(a)g(a) - f(b)g(a) + f(a)g(a) \\ &= f(a)g(b) - f(b)g(a) \\ h(b) &= f(b)g(b) - f(b)g(a) - f(b)g(b) + f(a)g(b) = f(a)g(b) - f(b)g(a) \\ &= f(a)g(b) - f(b)g(a) \\ &= h(a) \end{aligned}$$

So $h(a) = h(b)$, and, by Rolle's Theorem, there exists a point $c \in (a{,}b)$ such that $h'(c) = 0$, and

$$\begin{aligned} h'(x) &= f'(x)[g(b) - g(a)] - g'(x)[f(b) - f(a)] \\ h'(c) &= f'(c)[g(b) - g(a)] - g'(c)[f(b) - f(a)] \\ &= 0 \end{aligned}$$

so $f'(c)[g(b) - g(a)] = g'(c)[f(b) - f(a)]$. If $g(a) = g(b)$, then a stationary point of g would exist in $(a{,}b)$ by Rolle's Theorem, but g' is given to be non-zero over $(a{,}b)$, so $g(a) \neq g(b)$. We can then divide both sides of the equation by $g'(c)[g(b) - g(a)]$, obtaining the result. ■

l'Hôpital's Rule: If $f{,}g : I \to \mathbb{R}$ are differentiable on the open interval I and $f(c) = g(c) = 0$ at some point $c \in I$, then

$$\lim_{x\to c} \frac{f(x)}{g(x)} = \lim_{x\to c} \frac{f'(x)}{g'(x)}$$

provided the second limit exists.

Note: You cannot use l'Hôpital's rule on expressions like $\lim_{x\to 0} \frac{\sin(x)}{x}$, as l'Hôpital's rule relies on derivatives, and the limit in question is required to know the derivative of sin. In these cases, you should use the sandwich theorem.

Proof. Suppose $f{,}g$ are differentiable at $c \in \mathbb{R}$, $f'{,}g'$ are continuous, $f(c) = g(c) = 0$ and $g'(c) \neq 0$. Then,

$$\lim_{x\to c} \frac{f(x)}{g(x)} = \lim_{x\to c} \frac{f(x) - 0}{g(x) - 0}$$

$$
\begin{aligned}
&= \lim_{x \to c} \frac{f(x) - f(c)}{g(x) - g(c)} \\
&= \lim_{x \to c} \frac{\left(\frac{f(x)-f(c)}{x-c}\right)}{\left(\frac{g(x)-g(c)}{x-c}\right)} \\
&= \frac{\lim_{x \to c} \left(\frac{f(x)-f(c)}{x-c}\right)}{\lim_{x \to c} \left(\frac{g(x)-g(c)}{x-c}\right)} \\
&= \frac{f'(c)}{g'(c)} \\
&= \lim_{x \to c} \frac{f'(x)}{g'(x)}
\end{aligned}
$$

■

2.12.1 Taylor's Theorem with Remainders

Taylor's Theorem with Lagrange Remainder: If $f : I \to \mathbb{R}$ is n times differentiable on the open interval I, and $x,a \in I$, then,

$$f(x) = f(a) + f'(a)(x-a) + \frac{f''(a)}{2}(x-a)^2 + \cdots + \frac{f^{(n-1)}(a)}{(n-1)!}(x-a)^{n-1} + \frac{f^{(n)}(t)}{n!}(x-a)^n$$

for some $t \in (x,a)$.

Taylor's Theorem with Cauchy Remainder: If $f : I \to \mathbb{R}$ is n times differentiable on the open interval I, and $x,a \in I$, then,

$$f(x) = f(a) + f'(a)(x-a) + \frac{f''(a)}{2}(x-a)^2 + \cdots + \frac{f^{(n-1)}(a)}{(n-1)!}(x-a)^{n-1} + \frac{f^{(n)}(t)}{(n-1)!}(x-t)^{n-1}(x-a)$$

for some $t \in (x,a)$.

Proof. The function,

$$g(x) = f(x) - \left(f(a) + f'(a)(x-a) + \frac{f''(a)}{2}(x-a)^2 + \cdots + \frac{f^{(n-1)}(a)}{(n-1)!}(x-a)^{n-1}\right)$$

satisfies $g(a) = 0$, $g'(a) = 0$, $\cdots$, $g^{(n-1)}(a) = 0$, and $g^{(n)}(x) = f^{(n)}(x)$, because the bracket on the RHS goes to 0 after n differentiations.

If we let,

$$h(x) = g(x) - g(b)\frac{(x-a)^n}{(b-a)^n}$$

then the first $(n-1)$ derivatives of h also vanish at $x = a$, but we also have $h(b) = 0$.

Now, we proceed inductively. Since $h(n) = h(a) = 0$, there exists a point $t_1 \in (a,b)$ such that $h'(t_1) = 0$ by Rolle's Theorem. Since $h'(t_1) = h(a) = 0$, we again apply Rolle's Theorem so a there exists a point $t_2 \in (a,t_1)$ such that $h''(t_2) = 0$. Repeating this process, we eventually find a point $t = t_n$ where $h^{(n)}(t) = 0$.

$$h^{(n)}(t) = g^{(n)}(t) - g(b)\frac{n!}{(b-a)^n}$$

$$= 0$$

So,

$$g^{(n)}(t) = g(b)\frac{n!}{(b-a)^n}$$
$$g(b) = \frac{g^{(n)}(t)}{n!}(b-a)^n$$

Recalling our definition of $g(x)$, we have,

$$= \frac{f^{(n)}(t)}{n!}(b-a)^n$$

■

Proof. Let G be continuous over $[a,x]$ and differentiable over (a,x) with $G' \neq 0$, and let

$$F(t) = f(t) + f'(t)(x-t) + \frac{f''(t)}{2!}(x-t)^2 + \cdots \frac{f^{(n)}(t)}{n!}(x-t)^n$$

for $t \in [a,x]$. Note that $F(x) = f(x)$ due to every term after the first having $(x-t)$ as a factor. Then, by Cauchy's MVT, there exists some $c \in (a,x)$ such that,

$$\frac{F'(c)}{G'(c)} = \frac{F(x) - F(a)}{G(x) - G(a)} \qquad (\star)$$

Note that $F(x) - F(a) = R_n(x)$ is exactly the remainder term of the Taylor polynomial of $f(x)$.

$$\begin{aligned} F'(t) =& f'(t) + [f''(t)(x-t) - f'(t)] + \left[\frac{f^{(3)}(t)}{2!}(x-t)^2 - \frac{f^{(2)}(t)}{1!}(x-t)\right] + \cdots \\ &+ \left[\frac{f^{(n+1)}(t)}{n!}(x-t)^n - \frac{f^{(n)}(t)}{(n-1)!}(x-t)^(n-1)\right] \\ =& \frac{f^{(n+1)}(t)}{n!}(x-t)^n \end{aligned}$$

Substituting the above into $(\star)$, we have,

$$R_n(x) = \frac{f^{(n+1)}(c)}{n!}(c-x)^n \frac{G(x) - G(a)}{G'(c)}$$

If we let $G(t) = (x-t)^{k+1}$, we get the Lagrange form of the remainder. If we let $G(t) = t - a$, we get the Cauchy form of the remainder. ■

3 List of Results & Definitions with Proofs Omitted

3.1 Analysis I

A sequence (a_n) *tends to infinity* if for every $C > 0$, there exists $N \geq 1$ such that $a_n > C$ for all $n > N$.

A sequence, (a_n) *converges to* or *tends to* a if $\forall\, \epsilon > 0\, \exists\, N \in \mathbb{N}$ such that $|a_n - a| < \epsilon\, \forall\, n > N$.

Bolzano-Weierstrass Theorem: Every bounded sequence has a convergent subsequence.

A sequence, (a_n), has the *Cauchy* property if $\forall\, \epsilon > 0\, \exists\, N \in \mathbb{N}$ such that $|a_n - a_m| < \epsilon\, \forall\, n,m > N$.

A sequence is *strictly contracting* if for some number $0 < l < 1$ (the *contraction factor*), $|a_{n+1} - a_n| \leq l|a_n - a_{n-1}|$ for all n.

Boundedness Condition: Suppose $a_n \geq 0$ for all n. Then, $\sum_{n=1}^{\infty} a_n$ converges if and only if the sequence of partial sums $(s_n) = \sum_{i=1}^{n} a_i$ is bounded.

Comparison Test: Suppose $0 \leq a_n \leq b_n$ for all n. If $\sum_{n=1}^{\infty} b_n$ converges, then $\sum_{n=1}^{\infty} a_n$ converges and $\sum_{n=1}^{\infty} a_n \leq \sum_{n=1}^{\infty} b_n$

Comparison Test: Suppose $0 \leq a_n \leq b_n$ for all n. If $\sum_{n=1}^{\infty} a_n$ diverges, then $\sum_{n=1}^{\infty} b_n$ diverges.

Ratio Test: Suppose $a_n > 0$ for all n, and $\frac{a_{n+1}}{a_n} \to l$. Then, $\sum_{n=1}^{\infty} a_n$ converges if $0 \leq l < 1$, and diverges if $l > 1$. If $l = 1$, the ratio test is inconclusive.

$$\int_{m}^{n+1} f(x)dx \leq \sum_{k=m}^{n} f(k) \leq \int_{m-1}^{n} f(x)dx$$

Integral Test for Convergence: Suppose the function $f : [1,\infty) \to \mathbb{R}$ is non-negative and decreasing. Then $\sum_{n=1}^{\infty} f(n)$ converges if and only if the increasing sequence $(\int_1^n f(x)dx)$ is bounded, or equivalently, the sequence $(\int_1^n f(x)dx)$ is convergent.

Integral Test for Divergence: Suppose the function $f : [1,\infty) \to \mathbb{R}$ is non-negative and decreasing. Then $\sum_{n=1}^{\infty} f(n)$ diverges if and only if the increasing sequence $(\int_1^n f(x)dx)$ is unbounded, or equivalently, the sequence $(\int_1^n f(x)dx)$ is divergent.

The series $\sum_{n=1}^{\infty} \frac{1}{n^p}$ converges for $p > 1$ and diverges for $0 < p \leq 1$.

Alternating Series Test: Suppose (a_n) is decreasing and null. Then, the alternating series $\sum_{n=1}^{\infty} (-1)^{n+1} a_n$ is convergent.

Ratio Test: Suppose $a_n \neq 0$ for all n and $\left|\frac{a_{n+1}}{a_n}\right| \to l$. Then, $\sum_{n=1}^{\infty} a_n$ converges absolutely (and hence converges) if $0 \leq l < 1$ and diverges if $l > 1$.

Ratio Test: Suppose $a_n \neq 0$ for all n and $\left|\frac{a_{n+1}}{a_n}\right| \to \infty$. Then, $\sum_{n=1}^{\infty} a_n$ diverges.

3.2 Analysis II

Continuity: A function $f : I \to R$ is said to be continuous at $c \in I$ if $\forall \epsilon > 0, \exists \delta > 0$, such that if $x \in I$ and $|x - c| < \delta$, then $|f(x) - f(c)| < \epsilon$.

Let $L^+ = \lim_{x \to c^+} f(x)$ and $L^- = \lim_{x \to c^-} f(x)$. Using these quantities, we define the three classes of discontinuities:

- A *removable discontinuity* is where $L^+ = L^- = L$, $f(c)$ exists, and $f(c) \neq L$. If $L^+ = L^-$ but $f(c)$ is undefined, then $f(c)$ is instead a *removable singularity.*
- A *jump discontinuity* is where $L^+ \neq L^-$. $f(c)$ can take any value.
- A *essential discontinuity* is where at least one of L^+ and L^- do not exist.

Sequential Continuity: Let $f : I \to \mathbb{R}$ and $c \in I$. Then, f is continuous at c if and only if, for every sequence (x_n) of points in I which converges to c, we have $f(x_n) \to f(c)$ as $n \to \infty$.

Intermediate Value Theorem: Let $f : [a,b] \to \mathbb{R}$ be continuous and suppose $f(a) < k < f(b)$. Then, there exists $c \in [a,b]$ such that $f(c) = k$

Continuous Image of an Interval: If $f : I \to \mathbb{R}$ is continuous over I, then the image of f is also an interval.

Existence of Inverses: Let $f : [a,b] \to \mathbb{R}$ be continuous and strictly increasing. Then f has a continuous inverse, f^{-1} defined over its image.

Extreme Value Theorem: Let $f : [a,b] \to \mathbb{R}$ be continuous over $[a,b]$. Then, there exists numbers $c,d \in [a,b]$ such that $f(c) \leq f(x) \leq f(d)$ for all $x \in [a,b]$.

Boundedness of Continuous Functions: If $f : [a,b] \to \mathbb{R}$ is continuous, then f is bounded.

Radius of Convergence I: Let $\sum_{n=0}^{\infty} a_n x^n$ be a power series with $\sum_{n=0}^{\infty} a_n t^n$ convergent. Then, $\sum_{n=0}^{\infty} a_n x^n$ converges absolutely for all x such that $|x| < |t|$.

Radius of Convergence II: Let $\sum_{n=0}^{\infty} a_n x^n$ be a power series. Then, one of the following statements holds:

- There is a positive real R such that the series converges if $|x| < R$ and diverges if $|x| > R$. If such a number exists, it is called the *radius of convergence.*
- The series converges only if $x = 0$. In this case, we say the radius of convergence is 0.
- The series converges for all real x. In this case, we say the radius of convergence is ∞.

Absolute Series: Let $\sum_{n=0}^{\infty} a_n x^n$ be a power series with radius of convergence R. Then, $\sum_{n=0}^{\infty} |a_n| x^n$also has radius of convergence R.

Limits of Functions: Let I be an open interval and f a real-valued function defined over I, except possibly at a point $c \in I$. We write,

$$\lim_{x \to c} f(x) = L$$

if for every $\epsilon > 0$, there exists $\delta > 0$ such that if $0 < |x - c| < \delta$, then $|f(x) - L| < \epsilon$.

Limits and Continuity: If $f : I \to \mathbb{R}$ is defined over the open interval I and $c \in I$, then f is continuous at c if and only if $\lim_{x \to c} f(x) = f(c)$.

Continuous and Sequential Limits: If $f : I \backslash \{c\} \to \mathbb{R}$ is defined over the interval $I \backslash \{c\}$, then $\lim_{x \to c} f(x) = L$ if and only if for every sequence (x_n) of points in $I \setminus \{c\}$ which converges to c, we have $f(x_n) \to L$.

Sandwich Theorem for Limits: Let I be an interval containing the point a. Let g,f,h be functions defined over I, except possibly at a. If for every $x \in I$, we have $g(x) \leq f(x) \leq h(x)$ and $\lim_{x \to a} g(x) = \lim_{x \to a} h(x) = L$, then $\lim_{x \to a} f(x) = L$.

Note that a does not have to lie within the interior of I, and can be an endpoint, with the limits above being evaluated as one-sided limits. Similarly, the statement holds for infinite intervals, where $x \to \pm\infty$.

Let $f : I \to \mathbb{R}$ and $c \in I$. f is *differentiable* at c if,

$$\lim_{h \to 0} \frac{f(c + h) - f(c)}{h}$$

exists. If so, we denote this limit $f'(c)$.

Letting $x = c + h$, we can equivalently write the derivative as,

$$\lim_{x \to c} \frac{f(x) - f(c)}{x - c}$$

Differentiability and Continuity: If $f : I \to \mathbb{R}$ is differentiable at $c \in I$ then f is continuous at c.

Rolle's Theorem: Suppose $f : [a,b] \to \mathbb{R}$ is continuous over $[a,b]$ and differentiable over (a,b), and that $f(a) = f(b)$. Then, there is a point $c \in (a,b)$ such that $f'(c) = 0$.

Mean Value Theorem: Suppose $f : [a,b] \to \mathbb{R}$ is continuous over $[a,b]$ and differentiable over (a,b). Then, there is a point $c \in (a,b)$ such that,

$$f'(c) = \frac{f(b) - f(a)}{b - a}$$

This theorem may be more useful in the form,

$$f(b) - f(a) = (b - a)f'(c)$$

Functions with Positive Derivative: If $f : I \to \mathbb{R}$ is differentiable over the open interval I, and $f'(x) > 0$ for all $x \in I$, then f is strictly increasing over I.

Functions with Zero Derivative: If $f : I \to \mathbb{R}$ is differentiable over the open interval I and $f'(x) = 0$ for all $x \in I$, then f is constant over I.

Extrema and Derivatives: Suppose $f : [a,b] \to \mathbb{R}$ is continuous and is differentiable over (a,b). Then, f attains its maximum and minimum at points within the open interval where $f' = 0$, or at one of the endpoints, a or b.

Derivatives of Inverses: Let $f : (a,b) \to \mathbb{R}$ be differentiable with positive derivative. Then, $g = f^{-1}$ is differentiable, and $g'(x) = \frac{1}{f'(g(x))}$.

Differentiability of Power Series I: Let $\sum_{n=0}^{\infty} a_n x^n$ be a power series with radius of convergence R. Then, the series $\sum_{n=0}^{\infty} n a_n x^{n-1}$ has the same radius of convergence.

Differentiability of Power Series II: Let $\sum_{n=0}^{\infty} a_n x^n$ be a power series with radius of convergence R. Then, the function, $x \mapsto \sum_{n=0}^{\infty} a_n x^n$ is continuous and differentiable over $(-R,R)$.

Cauchy's Mean Value Theorem: If $f,g : [a,b] \to \mathbb{R}$ are continuous and differentiable over (a,b), and $g'(t) \neq 0$ for $t \in (a,b)$, then there exists a point c such that,

$$\frac{f'(c)}{g'(c)} = \frac{f(b) - f(a)}{g(b) - g(a)}$$

l'Hôpital's Rule: If $f,g : I \to \mathbb{R}$ are differentiable on the open interval I and $f(c) = g(c) = 0$ at some point $c \in I$, then

$$\lim_{x \to c} \frac{f(x)}{g(x)} = \lim_{x \to c} \frac{f'(x)}{g'(x)}$$

provided the second limit exists.

Taylor's Theorem with Lagrange Remainder: If $f : I \to \mathbb{R}$ is n times differentiable on the open interval I, and $x,a \in I$, then,

$$f(x) = f(a) + f'(a)(x - a) + \frac{f''(a)}{2}(x - a)^2 + \cdots + \frac{f^{(n-1)}(a)}{(n-1)!}(x - a)^{n-1} + \frac{f^{(n)}(t)}{n!}(x - a)^n$$

for some $t \in (x,a)$.

Taylor's Theorem with Cauchy Remainder: If $f : I \to \mathbb{R}$ is n times differentiable on the open interval I, and $x,a \in I$, then,

$$f(x) = f(a) + f'(a)(x - a) + \frac{f''(a)}{2}(x - a)^2 + \cdots + \frac{f^{(n-1)}(a)}{(n-1)!}(x - a)^{n-1} + \frac{f^{(n)}(t)}{(n-1)!}(x - t)^{n-1}(x - a)$$

for some $t \in (x,a)$.